Deriving models of change with interpretable parameters: linear estimation with nonlinear inference

Psicostats – December 2023

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Manuscript + Package

Deriving models of change with interpretable parameters: linear estimation with nonlinear inference

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December 10, 2(

Abstract

In the modeling of change over time, there is
opmental theories advanced in substantive research
longitudinal analysis. That is, theory is understood

https://osf.io/preprints/

leni: Linear Estimation with Nonlinear Inference

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Paper Citation: McCormick (2023). Deriving models of change with interpretable parameters: linear estimation with nonlinear inference. *PsyArixv*. https://doi.org/10.31234/osf.io/r4vxb

The `leni` package provides a collection of tools for Linear Estimation with Nonlinear Inference (LENI) which involves estimating the parameters of nonlinear target functions as transformations of linear models. Includes support for regression and mixed-effects models (main function: `leni()`) which produces transformed parameter estimates, as well as structural equation models (main function: `leni_sem()`) which generates `lavaan` syntax (https://lavaan.ugent.be/) for fitting linearized SEMs.

Installation

Set up for CRAN installation of `leni` is underway.

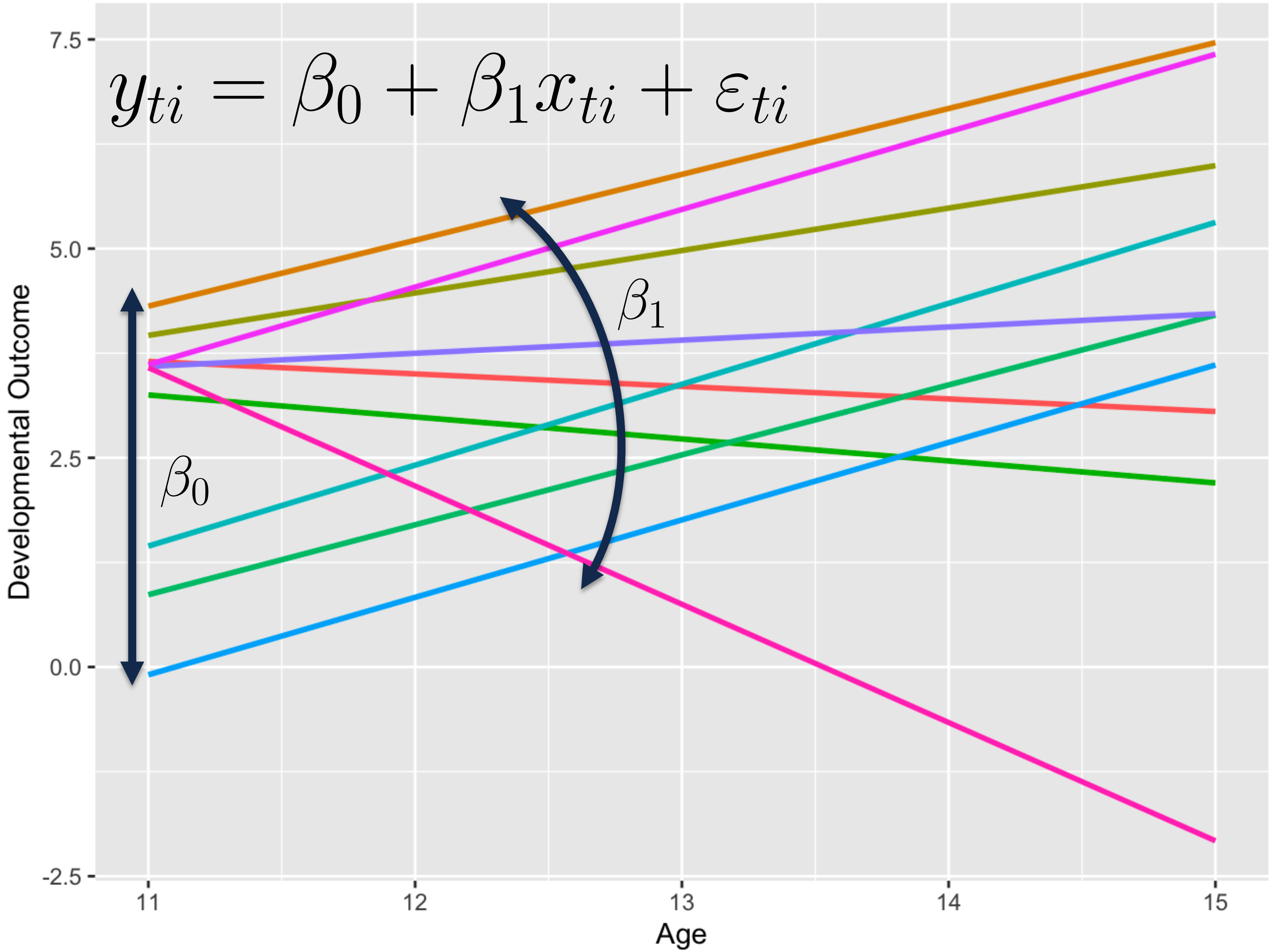
In the meantime, to install the latest development version, you can install `leni` directly from github using the `devtools` package.

```
#install.packages("devtools")
library(devtools)

install_github("E-M-McCormick/leni")
```

https://github.com/E-M-McCormick/leni

The case for meaningful parameters



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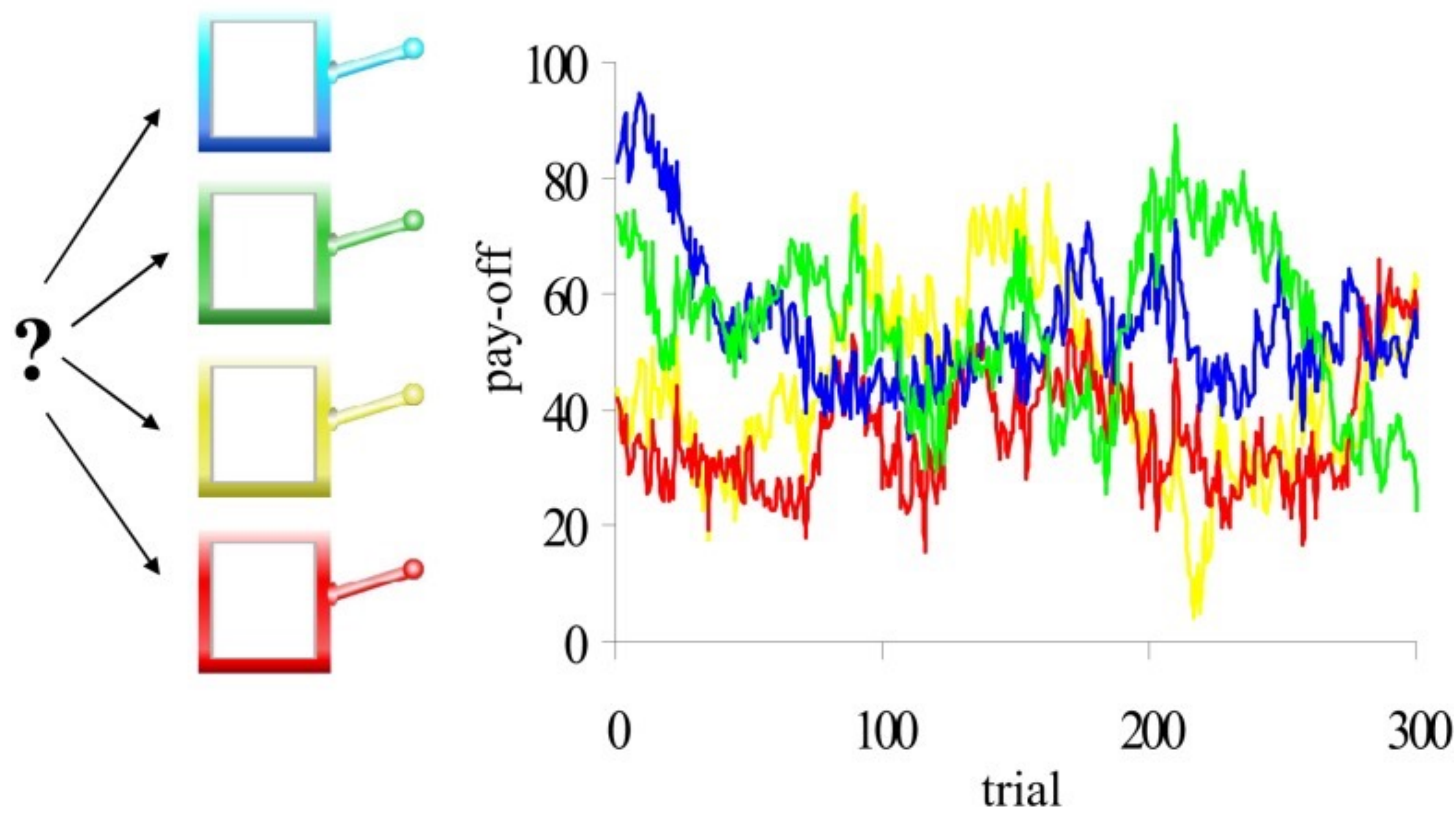

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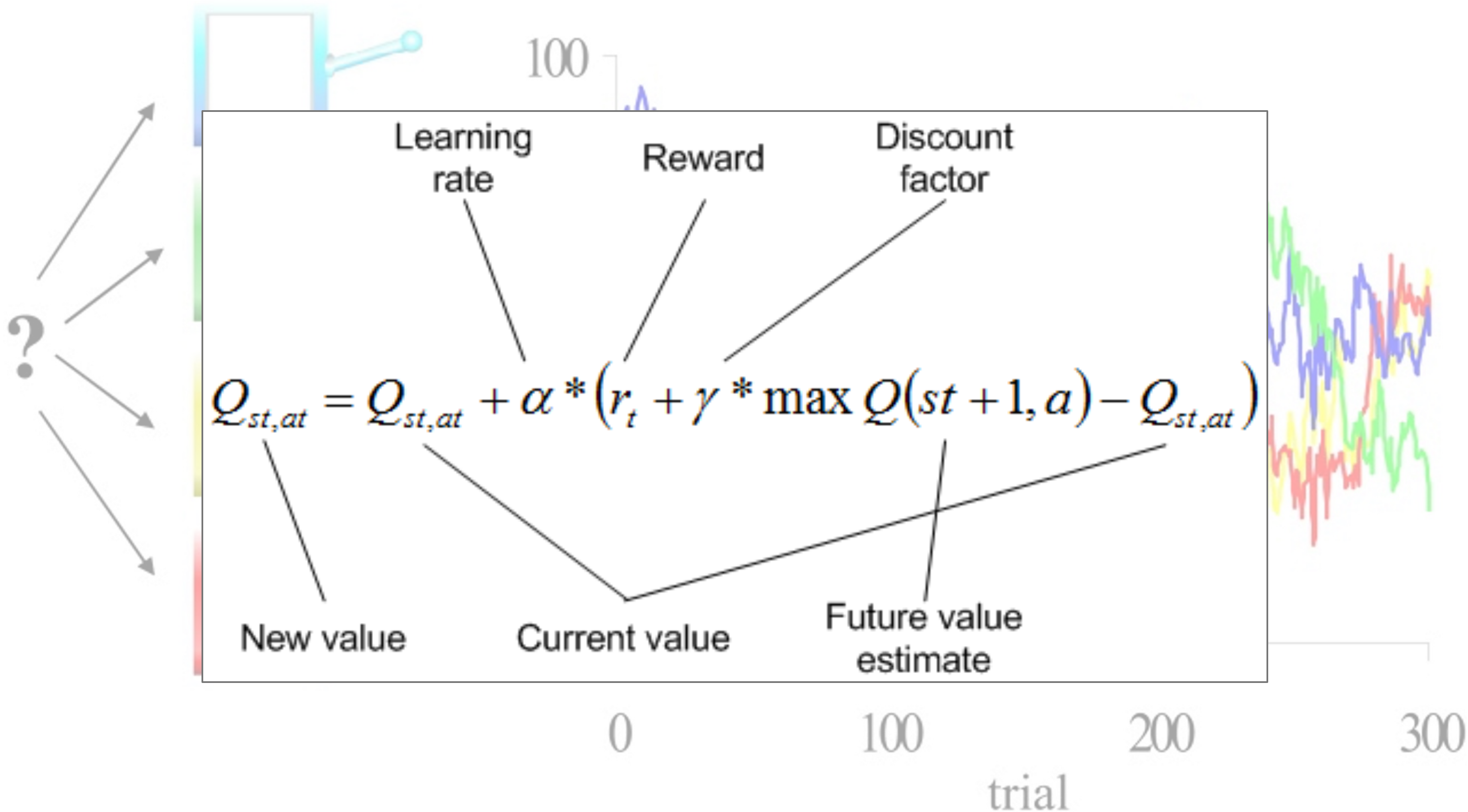

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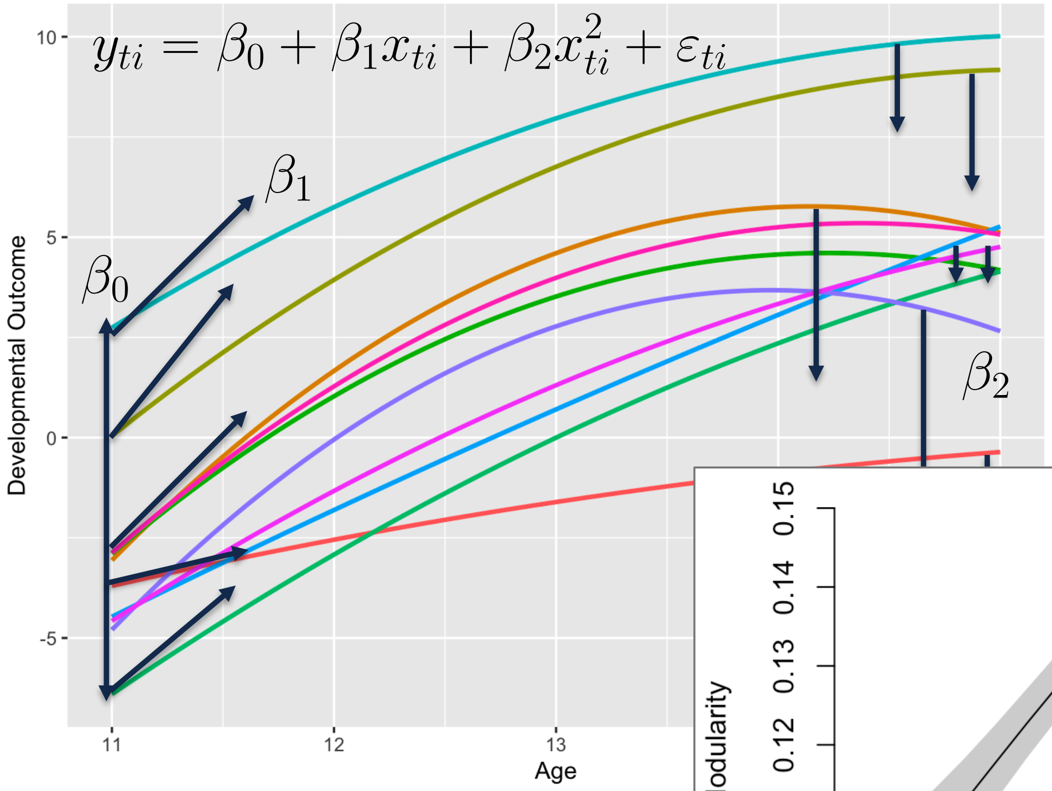
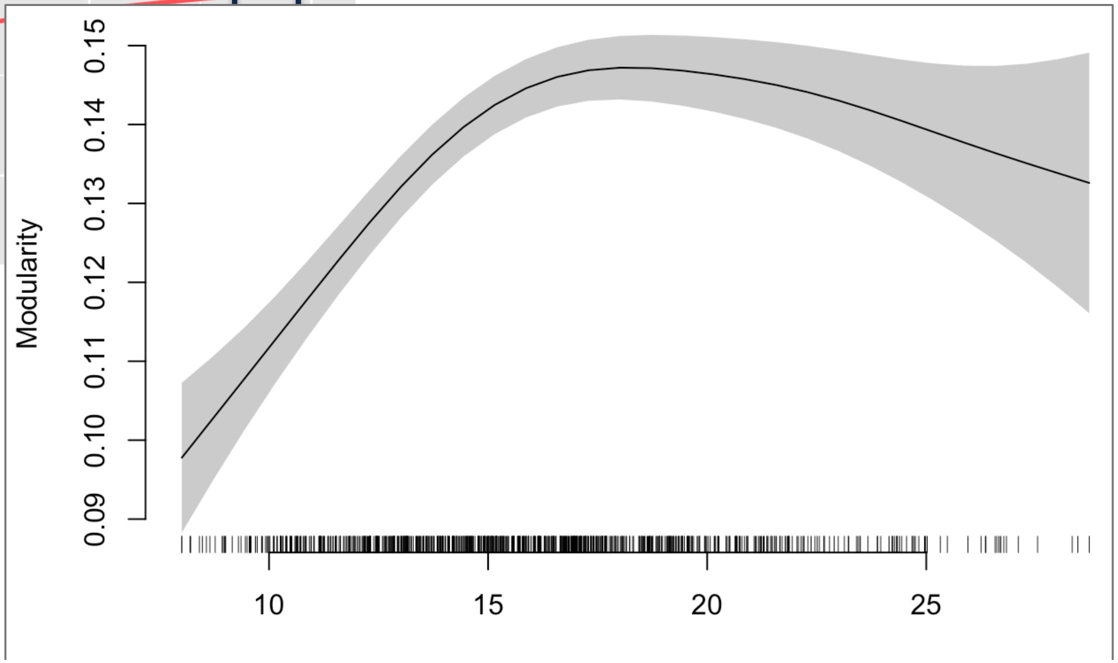
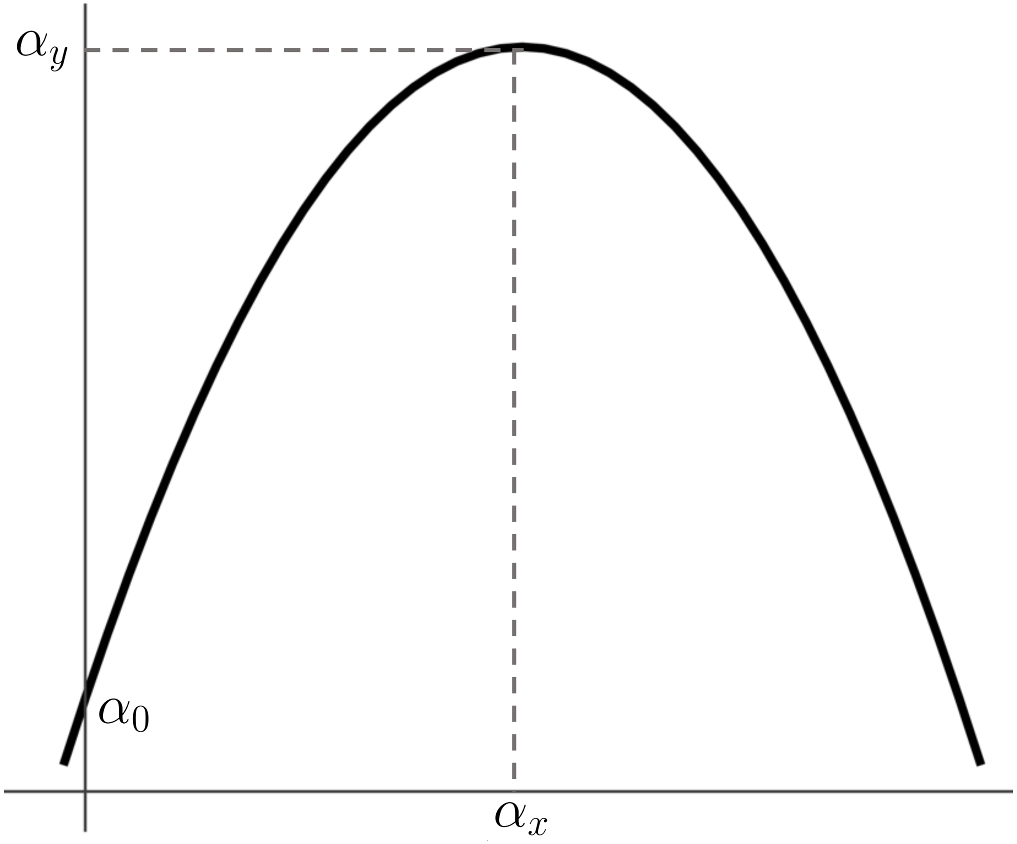




Why do we need interpretable parameters?

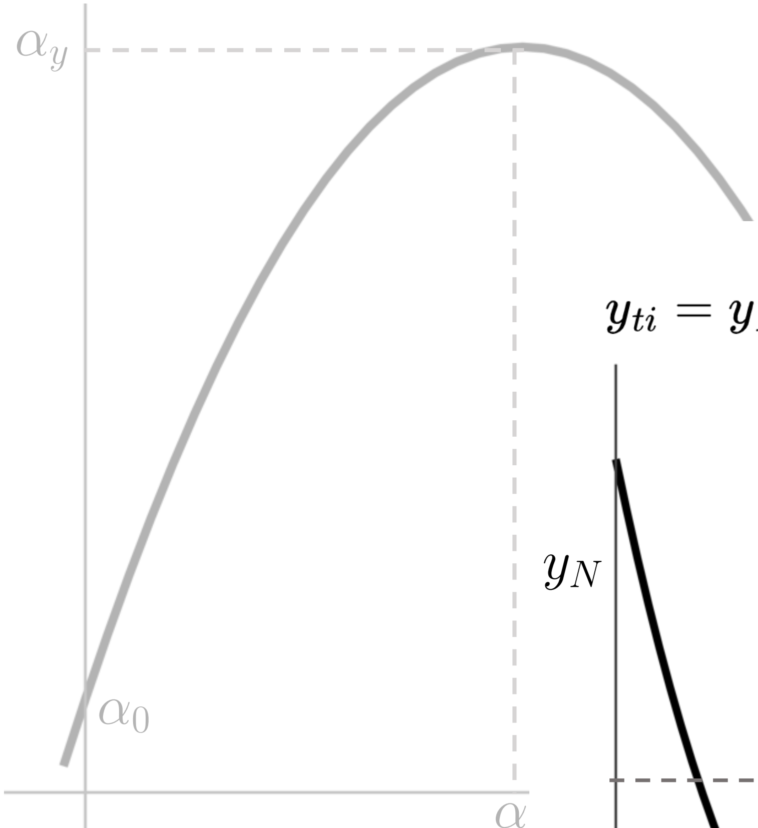
- Testing meaningful – and specifically articulated – theoretical hypotheses about change over time
  - Timing of inflections (e.g., peaks/troughs/plateaus), time-to-criterion, tempo of change
- Incorporating predictors of change
- Investigating distal outcomes associated with individual differences in change over time

Defining Nonlinear Equations



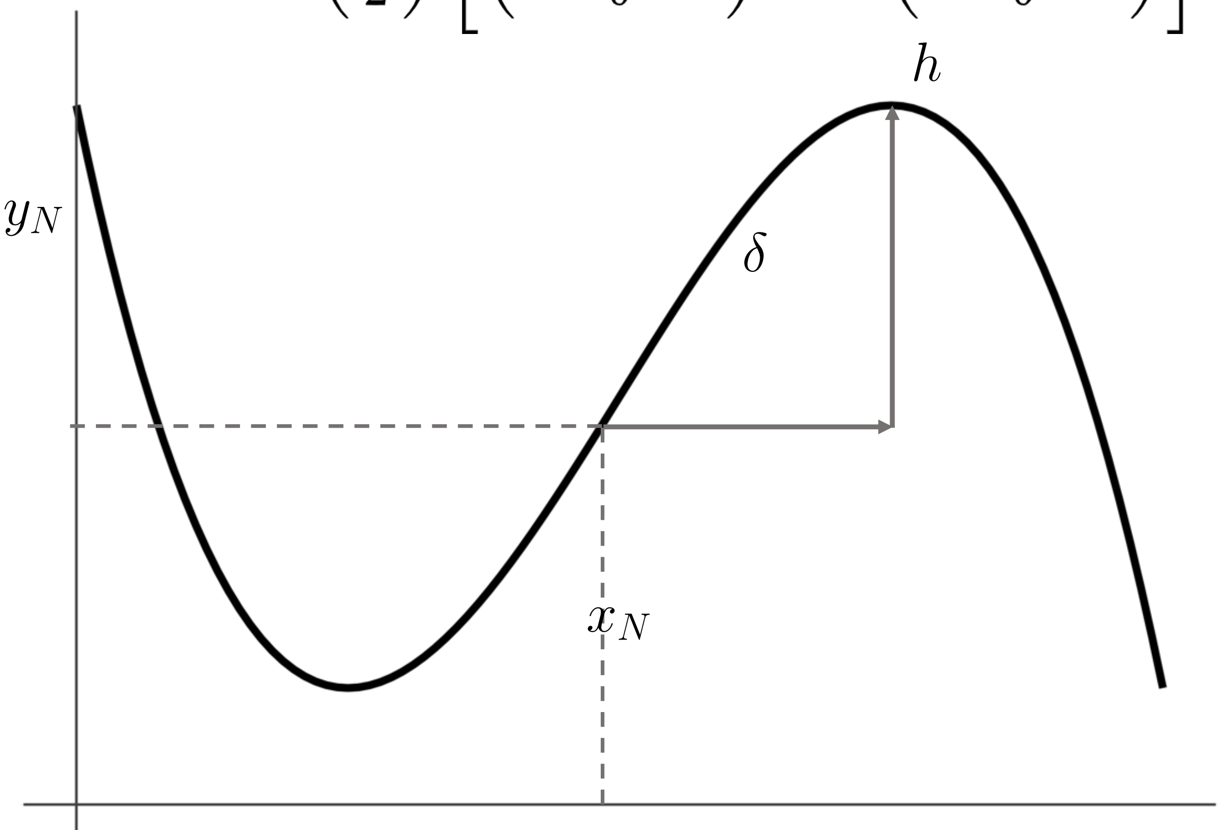
$$y_{ti} = \alpha_y - (\alpha_y - \alpha_0)\left(\frac{x_{ti}}{\alpha_x} - 1\right)^2 + \varepsilon_{ti}$$

Defining Nonlinear Equations

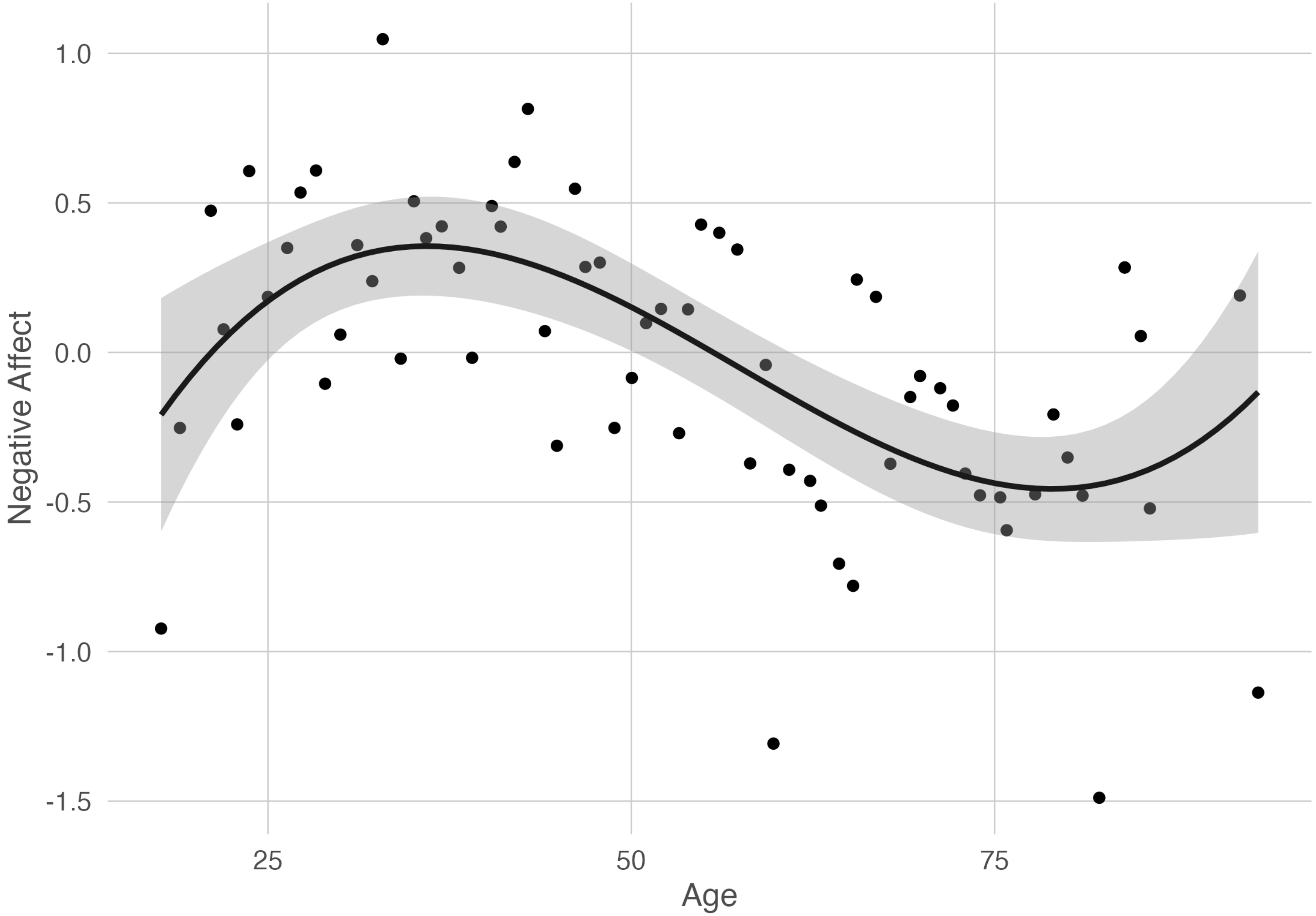


$y_{ti} = \alpha_y - (\alpha_y - \alpha$

$$y_{ti} = y_N - \left(\frac{h}{2}\right)\left[\left(\frac{x_{ti} - x_N}{\delta}\right)^3 - 3\left(\frac{x_{ti} - x_N}{\delta}\right)\right] + \varepsilon_{ti}$$



Defining Nonlinear Equations



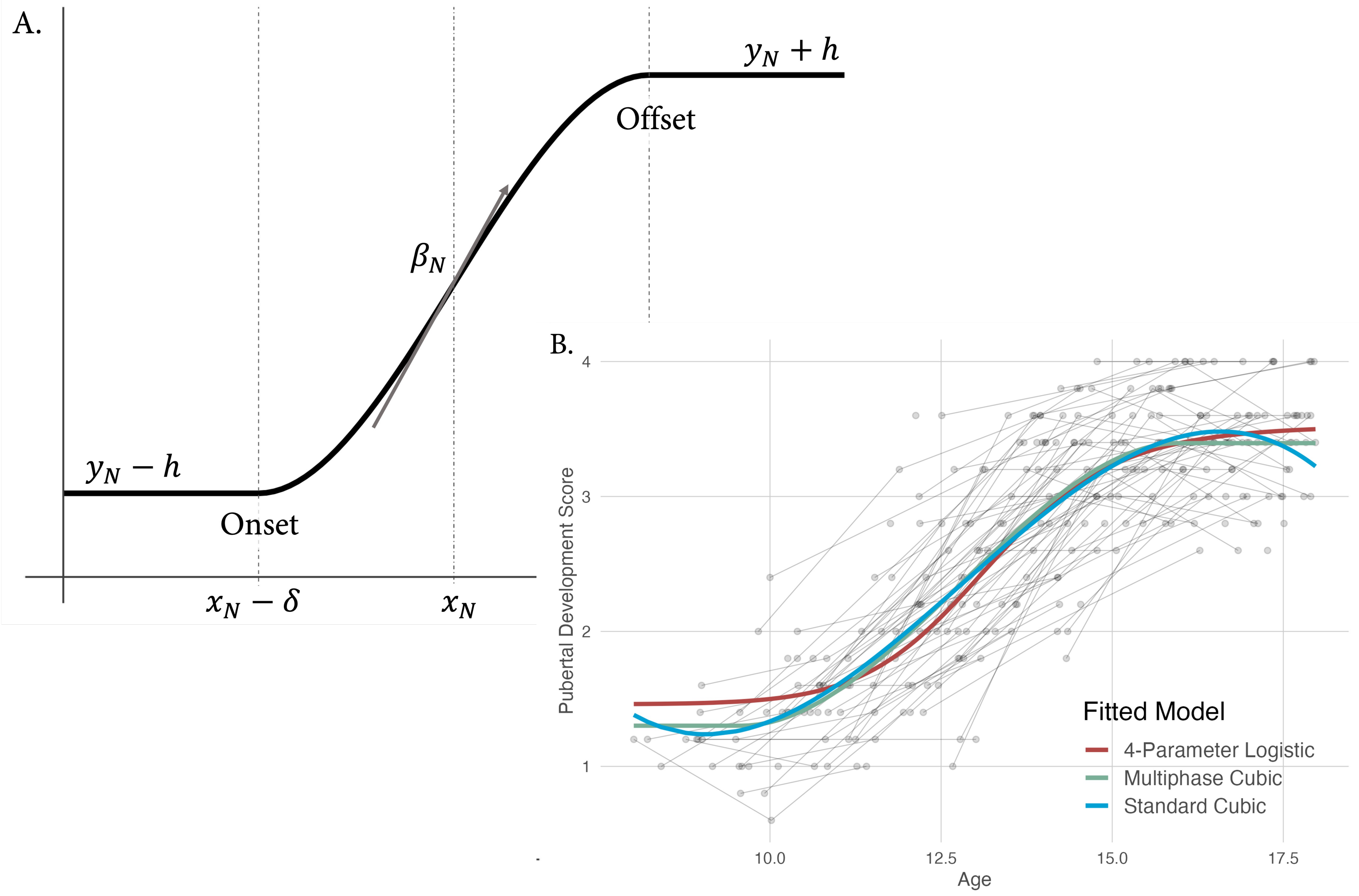
Defining Nonlinear Equations

Fitting Linear and Nonlinear Parameter Cubic Models

	Linear Parameter Model		Nonlinear Parameter Model
β_0	–2.288** (0.802)	x_N	57.428*** (2.162)
β_1	0.174** (0.052)	y_N	–0.050 (0.051)
β_2	–0.004*** (0.001)	δ	21.508*** (1.711)
β_3	2.04×10^{-5}** (6.10×10^{-6})	h	–0.406*** (0.068)
Num.Obs.	69		69
R^2	0.355		
AIC	76.6		76.6
BIC	87.7		87.7

Note: * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Defining Nonlinear Equations



Issues with Nonlinear Models

- Nonlinear with respect the parameters
  - Not defined for all values of all parameters

$$y_{ti} = \alpha_y - (\alpha_y - \alpha_0)\left(\frac{x_{ti}}{\alpha_x} - 1\right)^2 + \varepsilon_{ti}$$

$$y_{ti} = y_N - \left(\frac{h}{2}\right)\left[\left(\frac{x_{ti} - x_N}{\delta}\right)^3 - 3\left(\frac{x_{ti} - x_N}{\delta}\right)\right] + \varepsilon_{ti}$$

Issues with Nonlinear Models

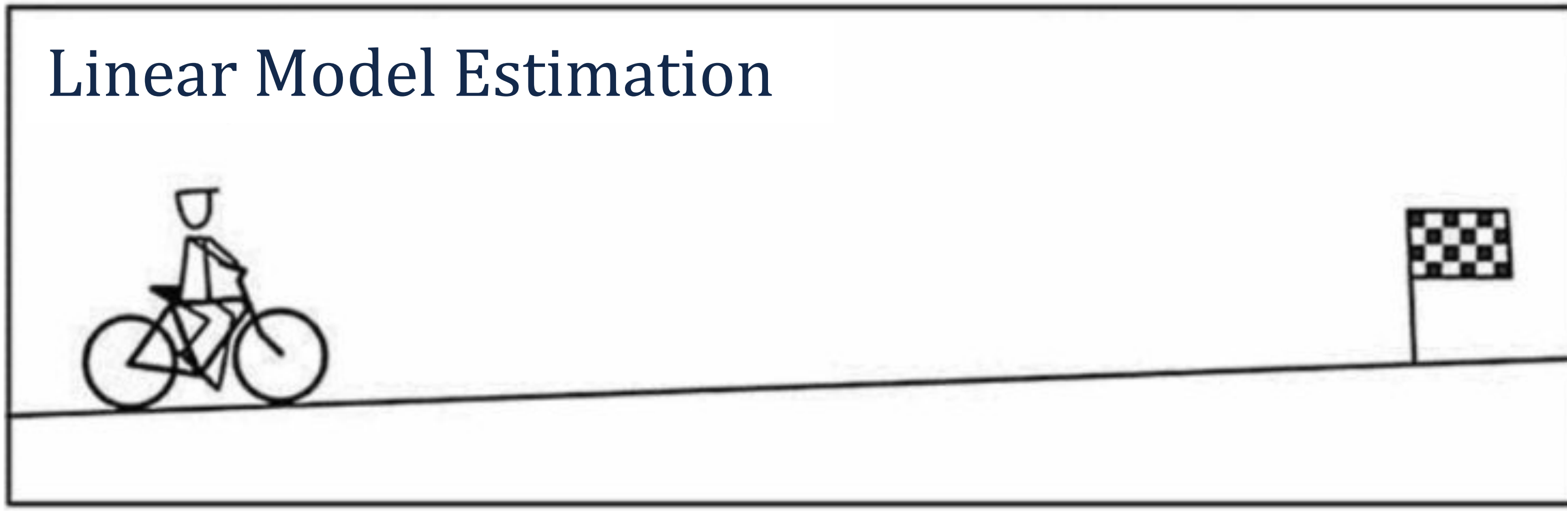
- No hierarchy of parameters for random effects

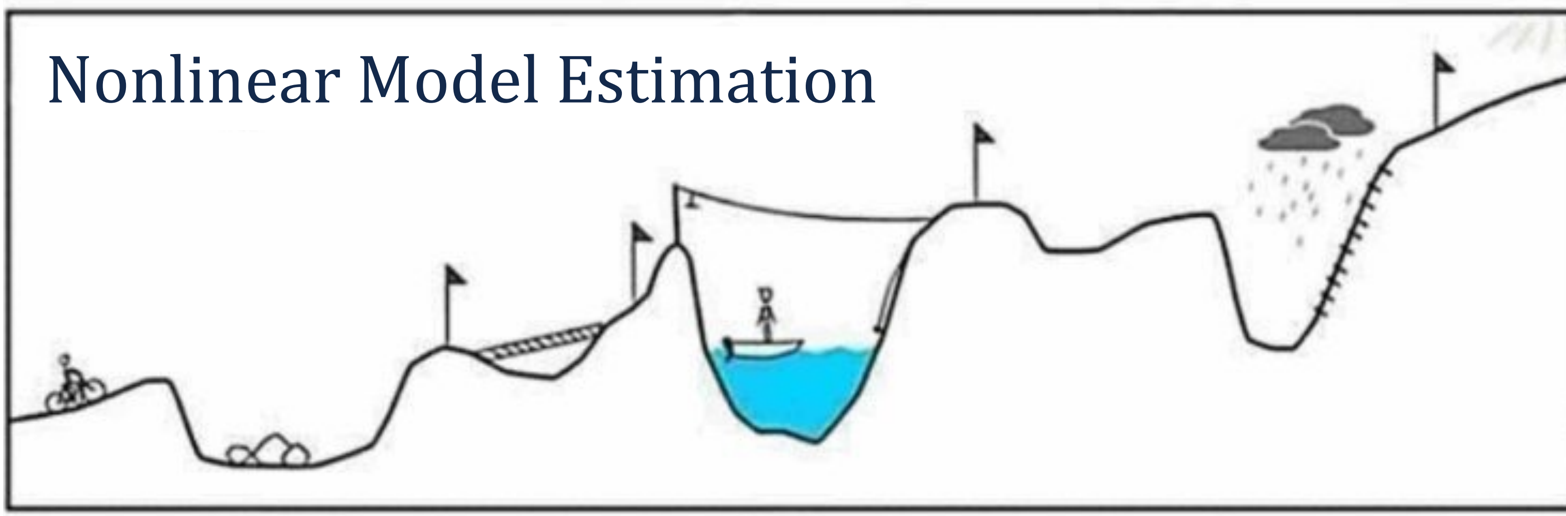
$$y_{ti} = \beta_0 + \beta_1 x_{ti} + \beta_2 x_{ti}^2 + \beta_3 x_{ti}^3 + \varepsilon_{ti}$$

$$y_{ti} = y_N - \left(\frac{h}{2}\right)\left[\left(\frac{x_{ti} - x_N}{\delta}\right)^3 - 3\left(\frac{x_{ti} - x_N}{\delta}\right)\right] + \varepsilon_{ti}$$

Issues with Nonlinear Models

- Nonlinear models are just harder to estimate





Solutions: Linear Estimation, Nonlinear Inference (LENI)

- Estimate the linear equation & define transformation functions to generate the implied nonlinear estimates

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<u>Fixed Effects + Standard Errors</u>

$$x_N = \frac{-\beta_2}{3\beta_3} \qquad y_N = \beta_0 - \frac{\beta_1\beta_2}{3\beta_3} + \frac{2\beta_2^3}{27\beta_3^2} \qquad \delta = \frac{\sqrt{\beta_2^2 - 3\beta_3\beta_1}}{3\beta_3} \quad h = -2\beta_3\delta^3$$

$$\text{ACOV}(f(x_N, y_N, \delta, h)) \approx \mathbf{J}'_{f(x_N,y_N,\delta,h)}\ \text{ACOV}(\boldsymbol{\beta})\ \mathbf{J}_{f(x_N,y_N,\delta,h)}$$

$$y_{ti} = y_N - \left(\frac{h}{2}\right)\left[\left(\frac{x_{ti} - x_N}{\delta}\right)^3 - 3\left(\frac{x_{ti} - x_N}{\delta}\right)\right] + \varepsilon_{ti}$$

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<u>Random Effects + Standard Errors</u>

$$\mathbf{T}_{f(x_N,y_N,\delta,h)} \approx \mathbf{J}'_{f(x_N,y_N,\delta,h)}\ \mathbf{T}_{\boldsymbol{\beta}}\ \mathbf{J}_{f(x_N,y_N,\delta,h)}$$

$$\mathrm{ACOV}(\mathbf{T}_{f(x_N,y_N,\delta,h)}) \approx \mathbf{J}'_{\mathbf{T}_{f(x_N,y_N,\delta,h)}}\ \mathrm{ACOV}(\mathbf{T}_{\boldsymbol{\beta}})\ \mathbf{J}_{\mathbf{T}_{f(x_N,y_N,\delta,h)}}$$

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Solutions: Linear Estimation, Nonlinear Inference (LENI)

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LENI Approach to Fixed Effects Estimation

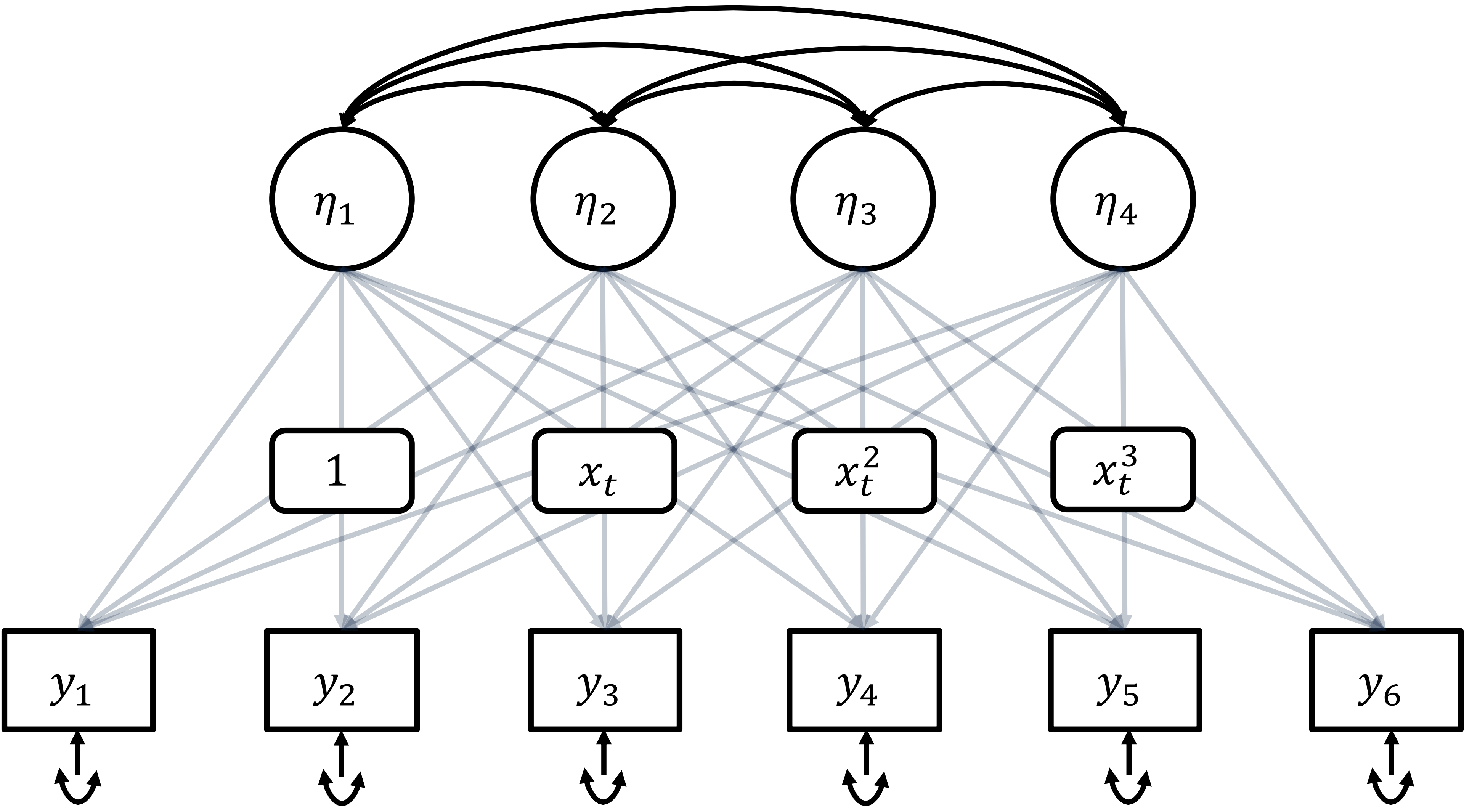
Pop. θ		Linear Estimates		LENI Estimates		Nonlinear Estimates	
Cubic Model							
x_N	0	β_0	9.994 (0.139)	x_N	−0.002 (0.116)	x_N	−0.002 (0.116)
y_N	10	β_1	−1.000 (0.080)	y_N	9.996 (0.094)	y_N	9.996 (0.094)
δ	3	β_2	2.25×10^{-4} (0.013)	δ	3.015 (0.107)	δ	3.015 (0.107)
h	−2	β_3	0.037 (0.005)	h	−2.010 (0.130)	h	−2.010 (0.130)
R^2	0.5		0.506				
BIC			921.90				921.90

Linearized SEM: Polynomials and Beyond

- LENI approach can be applied to mixed-effects polynomial models, but an linearized SEM approach can offer additional flexibility and target functions

Linearized SEM: Polynomials and Beyond

$$y_{ti} = \beta_0 + \beta_1 x_{ti} + \beta_2 x_{ti}^2 + \beta_3 x_{ti}^3 + \varepsilon_{ti}$$



Linearized SEM: Polynomials and Beyond

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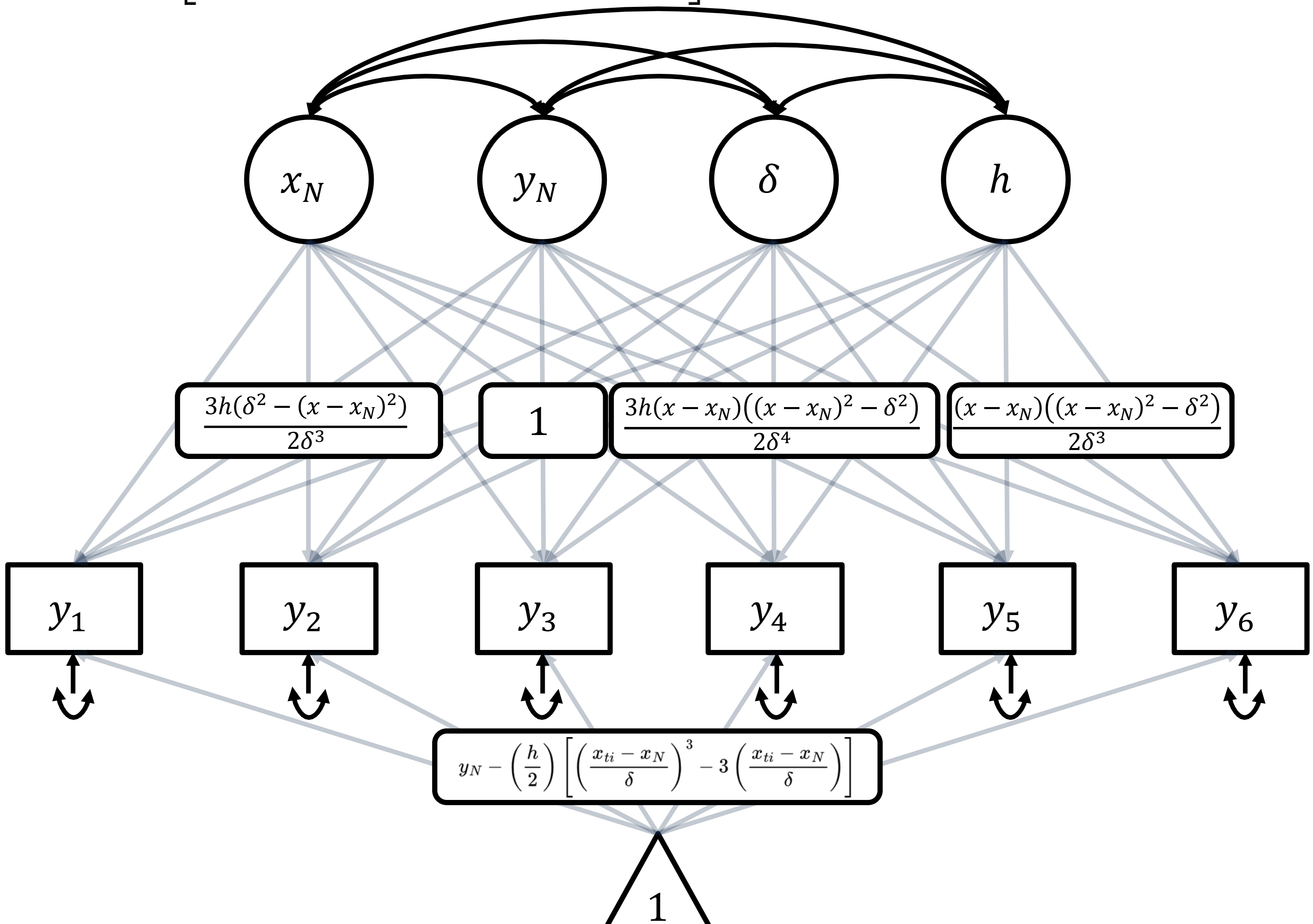
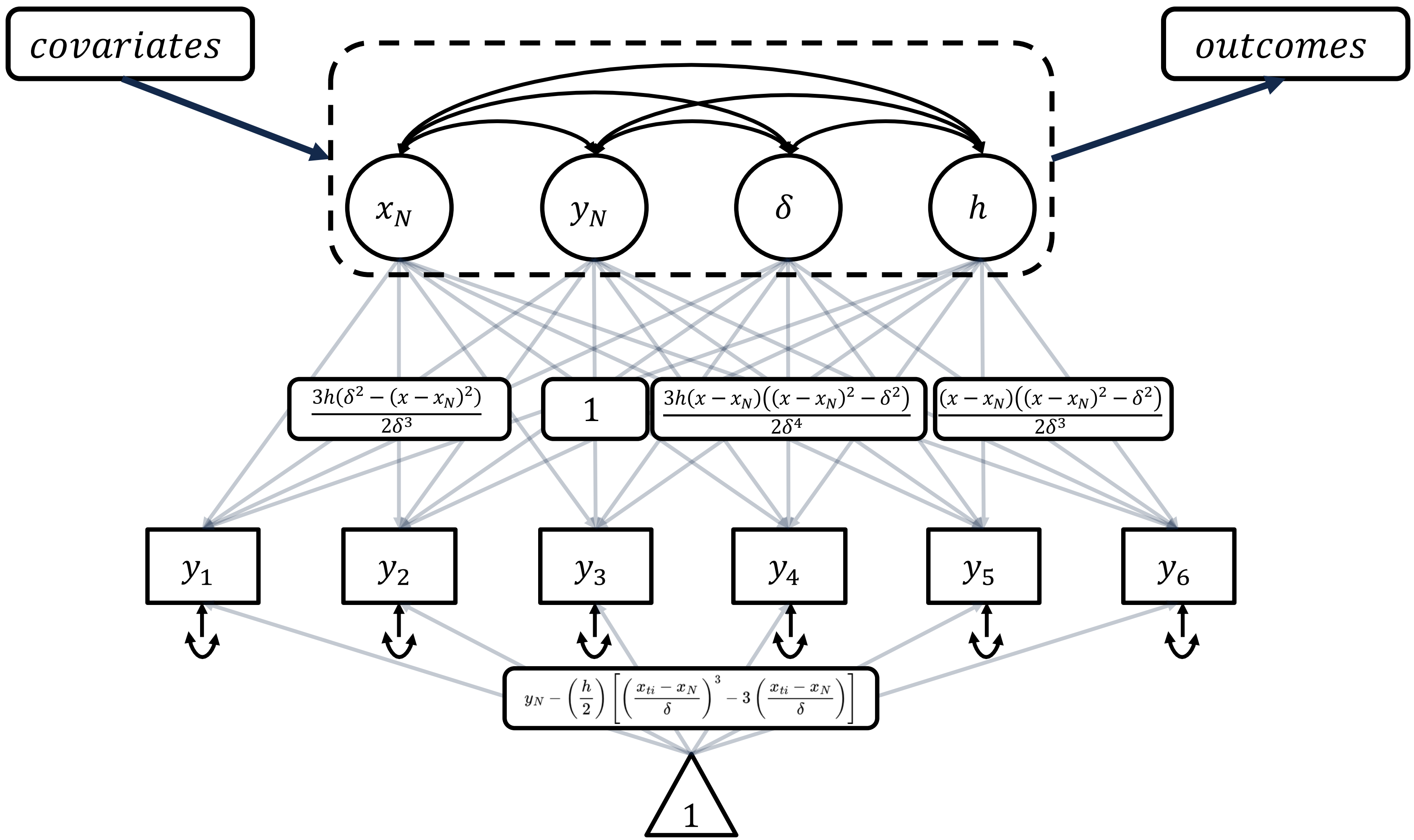
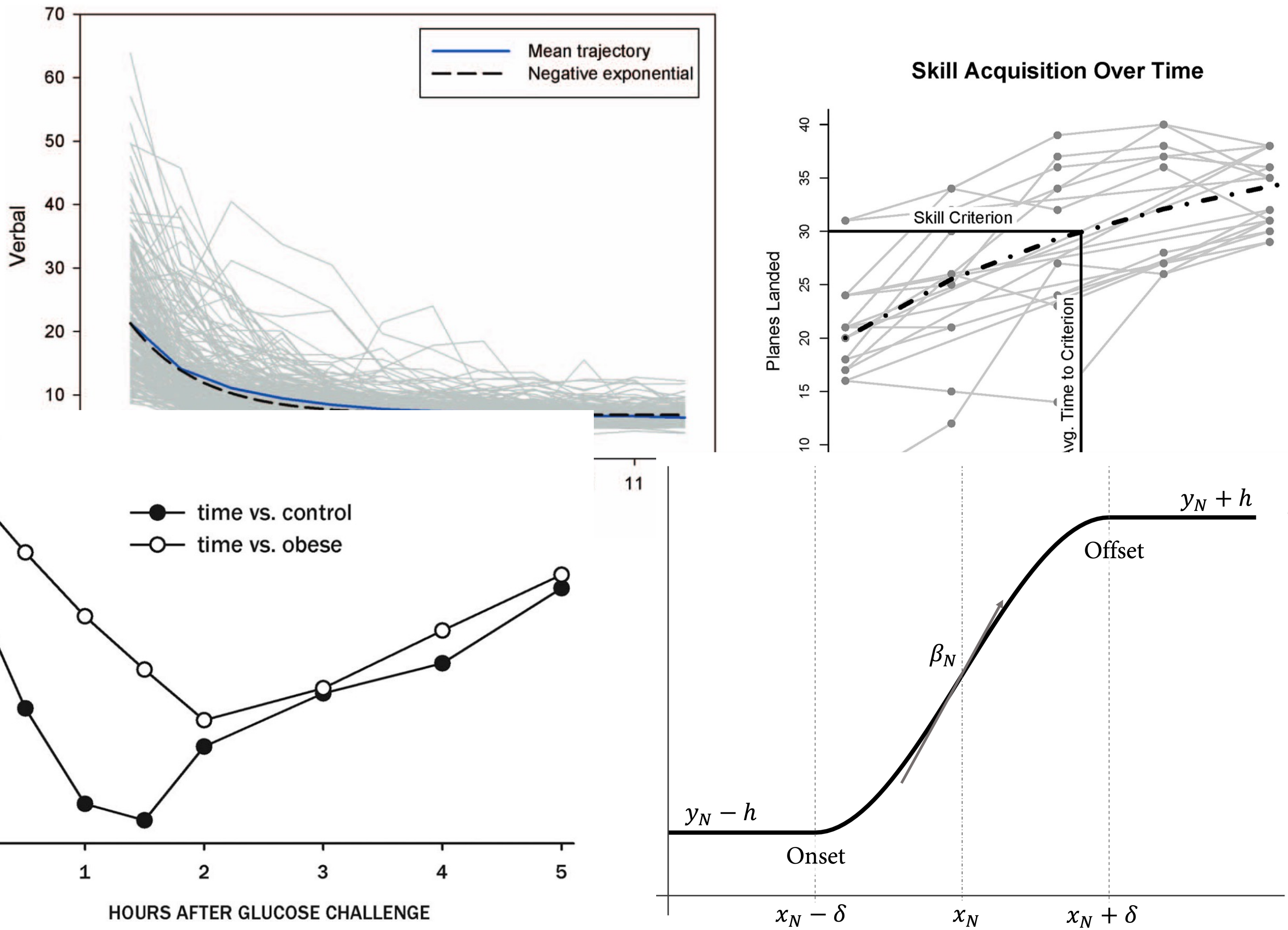


Linearized SEM: Polynomials and Beyond

- Easy inclusion of covariates and distal outcomes



Linearized SEM: Polynomials and Beyond



Moving forward with Meaningful Models

- Issues of estimation have limited the use of nonlinear models
  - Improvements in computation, LENI, linearized SEM offer new opportunities for using these models
  - Bayesian models for unstructured data

- Use models to test specific hypotheses and developmental features of interest
  - Identify new nonlinear target functions

- Meaningful parameters allow for tests of causes and consequences of individual differences in trajectories

Acknowledgements

- Collaborators
  - Patrick Curran, Ph.D.
  - Daniel Bauer, Ph.D.
  - Gregory Hancock, Ph.D.
  - Michelle Byrne, Ph.D.
  - John Flournoy, Ph.D.
  - Kathryn Mills, Ph.D.
  - Jennifer Pfeifer, Ph.D.
  - Rogier Kievit, Ph.D.
  - Sam Parsons, Ph.D.
  - Zsuzsa Bakk, Ph.D.

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